

Rational approximation to real points on plane algebraic curves

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A uniform exponent of approximation

Definition

For $\underline{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, set $\lambda(\underline{\xi}) = \sup$ of all $\lambda \geq 0$ such that

$$|x_0| \leq X, \quad |x_0\xi_1 - x_1| \leq X^{-\lambda}, \dots, |x_0\xi_n - x_n| \leq X^{-\lambda}$$

has a solution $\mathbf{x} = (x_0, x_1, \dots, x_n) \in \mathbb{Z}^{n+1} \setminus \{0\}$ for each $X \gg 1$.

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- $\lambda(\underline{\xi}) = 1/n$ for a.a. $\underline{\xi} \in \mathbb{R}^n$

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 $\Rightarrow \lambda(\underline{\xi}) \geq 1/(s-1)$ where $s = \dim_{\mathbb{Q}} \langle 1, \xi_1, \dots, \xi_n \rangle_{\mathbb{Q}}$
- $\lambda(\underline{\xi}) = 1/n$ for a.a. $\underline{\xi} \in \mathbb{R}^n$
- $\lambda(\underline{\xi}) \leq 1$ if $\underline{\xi} \notin \mathbb{Q}^n$
 $\Rightarrow \lambda(\xi) = 1$ if $\xi \notin \mathbb{Q}$ (case $n = 1$)

Let $\gamma := (1 + \sqrt{5})/2 = 1 + 1/\gamma = 1.618\dots$

Theorem (Davenport and Schmidt, 1969)

Let $\xi \in \mathbb{R}$ with $1, \xi, \xi^2, \dots, \xi^n$ linearly independent over $\mathbb{Q}$. Then

$$\lambda(\xi, \xi^2, \dots, \xi^n) \leq \lambda_n := \begin{cases} 1/\gamma \cong 0.618 & \text{if } n = 2, \\ 1/2 & \text{if } n = 3, \\ 1/\lfloor n/2 \rfloor & \text{if } n \geq 4. \end{cases}$$

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Corollary (Davenport and Schmidt, 1969)

Let ξ as above and let $\tau < 1 + (1/\lambda_n)$. There are infinitely many algebraic integers α of degree $\leq n + 1$ such that $|\xi - \alpha| \leq H(\alpha)^{-\tau}$ where $H(\alpha) = \|\text{Irr}_{\mathbb{Z}}(\alpha)\|$.

R. (2003): For $n = 2$, the above exponents are best possible.

Projective setting

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For $\Xi = (\xi_0 : \xi_1 : \dots : \xi_n) \in \mathbb{P}^n(\mathbb{R})$, set $\lambda(\Xi) = \sup$ of all $\lambda \geq 0$ such that

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Definition

Let $\mathcal{C}$ be a closed algebraic subset of $\mathbb{P}^n(\mathbb{R})$ of dimension 1, defined over $\mathbb{Q}$ and irreducible over $\mathbb{Q}$. We set

- $\mathcal{C}^{\text{li}} := \{\Xi \in \mathcal{C} ; \Xi \text{ has } \mathbb{Q}\text{-linearly independent coordinates}\}$
- $\lambda(\mathcal{C}) := \sup\{\lambda(\Xi) ; \Xi \in \mathcal{C}^{\text{li}}\}$, if $\mathcal{C}^{\text{li}} \neq \emptyset$.

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More precisely, let $\varphi \in \mathbb{Q}[x_0, x_1, x_2]$ be homogeneous of degree 2. Suppose that φ is irreducible over $\mathbb{Q}$ and that the set $\mathcal{C}$ of zeros of φ in $\mathbb{P}^2(\mathbb{R})$ is infinite. Then:

- (a) $\lambda(\Xi) \leq 1/\gamma$ for any $\Xi \in \mathcal{C}^{\text{li}}$,*
- (b) $\{\Xi \in \mathcal{C}^{\text{li}}; \lambda(\Xi) = 1/\gamma\}$ is countably infinite.*

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- (b) $\{\Xi \in \mathcal{C}^{\text{li}}; \lambda(\Xi) = 1/\gamma\}$ is countably infinite.

Example 1: Let $\varphi = x_0x_2 - x_1^2$, then

$$\mathcal{C}^{\text{li}} = \{(1 : \xi : \xi^2); \xi \in \mathbb{R}, [\mathbb{Q}(\xi) : \mathbb{Q}] > 2\},$$

and we recover

(a): Davenport and Schmidt (1969)

(b): R. (2003)

Example 2: Let $\varphi = 2x_0^2 - x_1^2$. Then,

$$\mathcal{C} = \{(1 : \pm\sqrt{2} : \xi) ; \xi \in \mathbb{R}\} \cup \{(0 : 0 : 1)\}$$

and $\lambda(\mathcal{C}) = 1/\gamma$.

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and $\lambda(\mathcal{C}) = 1/\gamma$. More precisely:

- Let $\xi \in \mathbb{R} \setminus \mathbb{Q}(\sqrt{2})$. There exists $c = c_1(\xi) > 0$ such that

$$(*) \quad |x_0| \leq X, \quad |x_0\sqrt{2} - x_1| \leq cX^{-1/\gamma}, \quad |x_0\xi - x_2| \leq cX^{-1/\gamma},$$

has no solution $(x_0, x_1, x_2) \in \mathbb{Z}^3 \setminus \{0\}$ for arbitrarily large X .

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- There exists $\xi \in \mathbb{R} \setminus \mathbb{Q}(\sqrt{2})$ and $c = c_2(\xi) > 0$ such that $(*)$ has a solution for each $X \geq 1$.

A particular cubic

Theorem (with S. Lozier)

For any $\xi \in \mathbb{R}$ such that $1, \xi, \xi^3$ are linearly independent over $\mathbb{Q}$, we have

$$\lambda(1 : \xi : \xi^3) \leq \frac{2(9 + \sqrt{11})}{35} = 0.7038\dots$$

i.e. the cubic $\mathcal{C} : x_0^2 x_2 - x_1^3 = 0$ in $\mathbb{P}^2(\mathbb{R})$ has $\lambda(\mathcal{C}) \leq 0.7038\dots$

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Problem: For a number field K of degree n over $\mathbb{Q}$, determine

$$\sup\{\lambda(\omega_1 : \dots : \omega_n : \xi) ; \xi \in \mathbb{R} \setminus K\},$$

where $(\omega_1, \dots, \omega_n)$ is a basis of K over $\mathbb{Q}$.

The sequence of minimal points

Let $1, \xi, \eta \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Put

$$\Xi = (1, \xi, \eta).$$

For each $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$, define

$$\|\mathbf{x}\| = \max_{0 \leq i \leq 2} |x_i|, \quad L(\mathbf{x}) = \max\{|x_0\xi - x_1|, |x_0\eta - x_2|\} = \|\mathbf{x} - x_0\Xi\|.$$

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There exist $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ in $\mathbb{Z}^3$ (called *minimal points*) such that

$$X_i := \|\mathbf{x}_i\| \quad \text{and} \quad L_i := L(\mathbf{x}_i)$$

satisfy

- $X_1 < X_2 < X_3 < \dots$,
- $L_1 > L_2 > L_3 > \dots$,
- if $\mathbf{x} \in \mathbb{Z}^3 \setminus \{0\}$ has $L(\mathbf{x}) < L_i$, then $\|\mathbf{x}\| \geq X_{i+1}$.

Facts

- For each $i \geq 1$, the subspace $W_i := \langle \mathbf{x}_i, \mathbf{x}_{i+1} \rangle_{\mathbb{R}}$ has dimension 2, $\{\mathbf{x}_i, \mathbf{x}_{i+1}\}$ is a basis of $W_i \cap \mathbb{Z}^2$, and

$$H(W_i) = \|\mathbf{x}_{i+1} \wedge \mathbf{x}_i\| \asymp X_{i+1} L_i.$$

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Let $I = \{i \geq 2; W_{i-1} \neq W_i\}$. For consecutive $i < j$ in I , we have

$$W_{i-1} \neq W_i = \cdots = W_{j-1} \neq W_j.$$

Then $X_j \asymp H(W_i \cap W_j) \ll H(W_i)H(W_j)$ and so

$$\boxed{X_j \ll (X_{i+1} L_i)(X_{j+1} L_j).}$$

Main hypothesis

Suppose that, for some fixed $\lambda > 1/2$, the conditions

$$\|\mathbf{x}\| \leq X \quad \text{and} \quad L(\mathbf{x}) \leq X^{-\lambda}$$

have a solution $\mathbf{x} \in \mathbb{Z}^3 \setminus \{0\}$ for each $X \gg 1$.

$$\implies \boxed{L_i \ll X_{i+1}^{-\lambda}} \quad \text{for each } i \geq 1.$$

In particular, for consecutive $i < j$ in I , we have

$$\boxed{X_j \ll X_{i+1}^{1-\lambda} X_{j+1}^{1-\lambda}}$$

$$\implies X_j^\lambda \ll X_{j+1}^{1-\lambda} \quad \text{since } X_{i+1} \leq X_j.$$

The case of conics

Suppose that $\varphi(1, \xi, \eta) = 0$ with $\varphi \in \mathbb{Q}[\mathbf{x}]_2$.

- $\varphi(\mathbf{x}_i) \neq 0$ for all $i \gg 1$
 $\Rightarrow 1 \ll |\varphi(\mathbf{x}_i)| \ll X_i L_i \ll X_i X_{i+1}^{-\lambda}$

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$$\Rightarrow (\mathbf{x}_i)_{i \in I} = (\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3, \dots) \quad \text{has}$$

$$\|\tilde{\mathbf{x}}_{i+1}\| \asymp \|\tilde{\mathbf{x}}_i\|^\gamma, \quad L(\tilde{\mathbf{x}}_i) \asymp \|\tilde{\mathbf{x}}_i\|^{-1},$$

$$|\varphi(\tilde{\mathbf{x}}_i)| \asymp 1, \quad |\det(\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_{i+1}, \tilde{\mathbf{x}}_{i+2})| \asymp 1.$$

A recurrence relation

Write $\varphi(\mathbf{x}) = \frac{1}{2}\Phi(\mathbf{x}, \mathbf{x})$ with Φ symmetric bilinear.

$$\varphi(p\mathbf{x} + q\mathbf{y}) = p^2\varphi(\mathbf{x}) + pq\Phi(\mathbf{x}, \mathbf{y}) + q^2\varphi(\mathbf{y})$$

So, $\psi(\mathbf{x}, \mathbf{y}) := \Phi(\mathbf{x}, \mathbf{y})\mathbf{x} - \varphi(\mathbf{x})\mathbf{y}$ has

$$\varphi(\psi(\mathbf{x}, \mathbf{y})) = \varphi(\mathbf{x})^2\varphi(\mathbf{y})$$

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We find that:

$$\|\psi(\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_{i+1})\| \ll \|\tilde{\mathbf{x}}_{i-2}\|$$

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Case of the cubic $\mathcal{C} : x_0^2 x_2 - x_1^3 = 0$

Suppose that $(1, \xi, \eta) \in \mathcal{C}$ i.e. that $\eta = \xi^3$. Put

$$\varphi(\mathbf{x}) = x_0^2 x_2 - x_1^3.$$

Proposition

Suppose that $\lambda > 2/3$. Then, for each $i \gg 1$, we have $\varphi(\mathbf{x}_i) \neq 0$ and so

$$1 \ll |\varphi(\mathbf{x}_i)| \ll X_i^2 L_i \ll X_i^2 X_{i+1}^{-\lambda}.$$

Corollary. $\lambda \leq \sqrt{3} - 1 \cong 0.732$

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Problem:

Find polynomial relations between $\mathbf{x}_i$ and $\mathbf{x}_j$ for consecutive $i < j$ in I without resorting to intermediate points, even when X_j/X_{i+1} is large.

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Proposition

Suppose that $\lambda > (5 - \sqrt{13})/2 \cong 0.697$, then for each $i \in I$ with $F_i \neq 0$, we have

$$1 \leq |F_i| \ll X_j^2 L_i^4 \ll X_j^2 X_{i+1}^{-4\lambda}$$

Corollary

If $F_i \neq 0$ for infinitely many $i \in I$, then $\lambda \leq 5/7 \cong 0.714$.

A graded subring

$$\mathcal{R} := \bigoplus_{\ell \geq 0} \mathcal{R}_\ell \quad \text{where} \quad \mathcal{R}_\ell = \mathbb{Q}[S, T, U, V] \cap \mathbb{Q}[\mathbf{x}, \mathbf{y}]_{(2\ell, \ell)}$$

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Proposition

- $T \in \mathcal{R}_1, \quad F \in \mathcal{R}_2, \quad S^2V \in \mathcal{R}_3$
- $\mathcal{R} = \mathbb{Q}[T, F, S^2V]$
- $\dim_{\mathbb{Q}} \mathcal{R}_\ell = \tau(\ell) := \#\{(m, n) \in \mathbb{N}^2; 2m + 3n \leq \ell\} \asymp \frac{\ell^2}{12}$

A valuation

For any bihomogeneous polynomial $P(\mathbf{x}, \mathbf{y}) \in \mathbb{Q}[\mathbf{x}, \mathbf{y}]$, define

$$\begin{aligned} v(P) &:= \text{largest integer } k \geq 0 \text{ such that } q^k | P(\mathbf{x}, \mathbf{x} + q\mathbf{y}) \text{ in } \mathbb{Q}[\mathbf{x}, \mathbf{y}, q] \\ &= \text{largest integer } k \geq 0 \text{ such that } q^k | P(\mathbf{x}, p\mathbf{x} + q\mathbf{y}) \text{ in } \mathbb{Q}[\mathbf{x}, \mathbf{y}, p, q] \end{aligned}$$

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Application: For consecutive $i < j$ in I , write

$$\mathbf{x}_j = p_i \mathbf{x}_i + q_i \mathbf{x}_{i+1} \quad \text{with} \quad p_i, q_i \in \mathbb{Z}.$$

- $\gcd(p_i, q_i) = 1, \quad |q_i| \asymp X_j/X_{i+1}$
- If $P(\mathbf{x}, \mathbf{y}) \in \mathcal{R}_\ell$ has $v(P) = k$, then

$$\begin{aligned} P(\mathbf{x}_i, \mathbf{x}_j) &= P(\mathbf{x}_i, p_i \mathbf{x}_i + q_i \mathbf{x}_{i+1}) = q_i^k \tilde{P}(\mathbf{x}_i, \mathbf{x}_{i+1}, p_i, q_i) \\ \implies P(\mathbf{x}_i, \mathbf{x}_j) &= 0 \quad \text{or} \quad |P(\mathbf{x}_i, \mathbf{x}_j)| \gg |q_i|^k \asymp \left(\frac{X_j}{X_{i+1}} \right)^k. \end{aligned}$$

The ideals $J^{(k)}$

Define $\rho: \mathbb{Q}[\mathbf{x}, \mathbf{y}] \longrightarrow \mathbb{Q}[\mathbf{x}, \mathbf{y}, q]$.

$$P(\mathbf{x}, \mathbf{y}) \longmapsto P(\mathbf{x}, \mathbf{x} + q\mathbf{y})$$

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- $J^{(k)} := \bigoplus_{\ell \geq 0} \{P \in \mathcal{R}_\ell; v(P) \geq k\}$ is an ideal of $\mathcal{R}$,
- $\dim_{\mathbb{Q}} \left(\frac{\mathcal{R}_\ell}{\mathcal{R}_\ell \cap J^{(k)}} \right) = \begin{cases} \tau(k-1) & \text{if } k \leq \ell, \\ \tau(\ell) & \text{if } k > \ell. \end{cases}$

Corollary

Suppose $\lambda > (1 + 3\sqrt{5})/11 \cong 0.7007$. Then there exists an integer $d \geq 1$ and $P \in \mathcal{R}_d \setminus \{0\}$ such that $P(\mathbf{x}_i, \mathbf{x}_j) = 0$ for infinitely many pairs of consecutive elements $i < j$ in I .

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Specific polynomials

$$F \in \mathcal{R}_2 \cap J^{(0)},$$

$$D^{(2)} := F^3 + 27(S^2V)^2 \in \mathcal{R}_6 \cap J^{(2)},$$

$$D^{(3)} := F^3 - 18TF(S^2V) - 135(S^2V)^2 \in \mathcal{R}_6 \cap J^{(3)}$$

$$D^{(6)} := \dots \in \mathcal{R}_9 \cap J^{(6)}.$$

If $\lambda > 0.6985$, then $F_i, D_i^{(2)}, D_i^{(3)}, D_i^{(6)} \neq 0$ for each large enough $i \in I$.

$$(\dots) \Rightarrow \lambda \leq \frac{2(9 + \sqrt{11})}{35} \cong 0.7038$$

Example: non-vanishing of F_i

$$\begin{aligned} A = T^2 - 3SU \in \mathcal{R}_2 \cap J^{(2)} &\Rightarrow q^2 | A(\mathbf{x}, p\mathbf{x} + q\mathbf{y}) \text{ in } \mathbb{Z}[\mathbf{x}, \mathbf{y}, p, q], \\ &\Rightarrow q_i^2 | A_i := A(\mathbf{x}_i, \mathbf{x}_j) \text{ in } \mathbb{Z}. \end{aligned}$$

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