

Parametric geometry of numbers

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Diophantine approximation and related fields

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I: Preliminaries

- $\mathbb{R}^n$ is an Euclidean space for the scalar product given by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$$

for each $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{R}^n$.

The canonical basis

$$\mathbf{e}_1 = (1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, \dots, 0, 1)$$

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- $\bigwedge^k \mathbb{R}^n$ is naturally an Euclidean space for each $k = 1, \dots, n$, the products

$$\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \quad (1 \leq i_1 < \cdots < i_k \leq n)$$

forming an orthonormal basis.

Let $\bigwedge^k \mathbb{Z}^n$ denote the lattice spanned by these products.

Geometry of numbers (Minkowski, 1889)

Let $\mathcal{C}$ be a convex body in $\mathbb{R}^n$

(i.e. a compact convex neighborhood $\mathcal{C}$ of 0 s.t. $\mathcal{C} = -\mathcal{C}$).

For $i = 1, \dots, n$, the i -th minimum of $\mathcal{C}$, denoted $\lambda_i = \lambda_i(\mathcal{C})$, is the smallest $\lambda > 0$ such that $\lambda\mathcal{C}$ contains at least i linearly independent elements of $\mathbb{Z}^n$.

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Theorem (Minkowski's second convex body theorem)

$$\frac{2^n}{n!} \leq \lambda_1 \cdots \lambda_n \text{vol}(\mathcal{C}) \leq 2^n.$$

Compound convex bodies (Mahler, 1955)

For $k = 1, \dots, n$ and $i = 1, \dots, N = \binom{n}{k}$, let

- $\mathcal{C}^{(k)}$ = convex hull of the products $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k$ with $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathcal{C}$.
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For $i = 1, \dots, N$, we have

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In particular,

- $\lambda_1^{(k)} \asymp \lambda_1 \cdots \lambda_k$,
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In practice, we use an approximation $\mathcal{C}^{(k)}$ of the k -th compound of $\mathcal{C}$.

Schmidt and Summerer theory (2009, 2013)

Let $\mathbf{u} \in \mathbb{R}^n$ with Euclidean norm $\|\mathbf{u}\| = 1$. Define

$$\mathcal{C}(Q) = \mathcal{C}_{\mathbf{u}}(Q) = \left\{ \mathbf{x} \in \mathbb{R}^n ; \|\mathbf{x}\| \leq 1, |\mathbf{x} \cdot \mathbf{u}| \leq \frac{1}{Q} \right\} \quad (Q \geq 1),$$

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Classical exponents of approximation can be expressed in terms of

$$\underline{\varphi}_i(\mathbf{u}) := \liminf_{q \rightarrow \infty} \frac{L_i(q)}{q} \quad \text{and} \quad \bar{\varphi}_i(\mathbf{u}) := \limsup_{q \rightarrow \infty} \frac{L_i(q)}{q} \quad i = 1, \dots, n.$$

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Using their theory, they recover

- Jarník's and Khintchine's transference principles,
- their refinements by Bugeaud, Laurent and Schmidt,
- constraints by Laurent (2009) on the 4 main exponents when $n = 3$.

Basic properties of $L_1, \dots, L_n$ (Schmidt, 1982)

$$\mathcal{C}(Q) = \{\mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| \leq 1, |\mathbf{x} \cdot \mathbf{u}| \leq Q^{-1}\}, \quad L_j(q) = \log \lambda_j(\mathcal{C}(e^q))$$

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1) $\boxed{L_1(q) \leq \dots \leq L_n(q)}$

This yields a map $\mathbf{L}_{\mathbf{u}}: [0, \infty) \longrightarrow \Delta_n$
 $q \longmapsto (L_1(q), \dots, L_n(q))$

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$$2) \quad \text{vol}(\mathcal{C}(Q)) \asymp Q^{-1} \Rightarrow \prod \lambda_i(\mathcal{C}(Q)) \asymp Q$$

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The trajectory of a point

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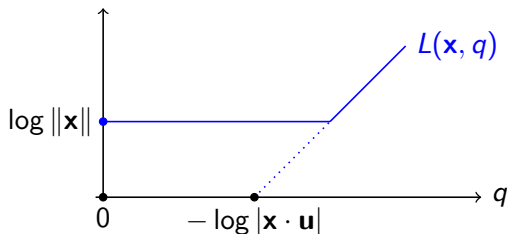
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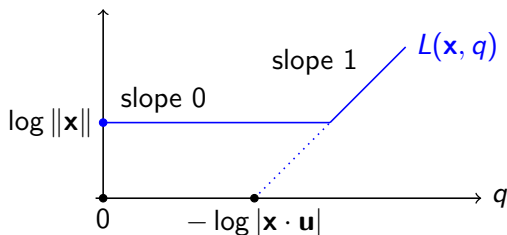


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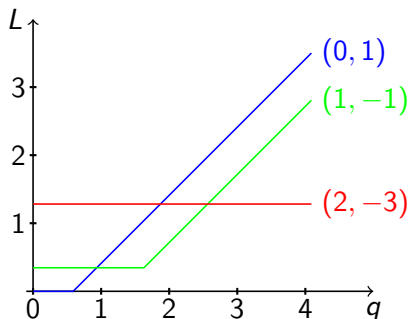
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Example: $n = 2, \mathbf{u} = (3, 2)/\sqrt{13}$

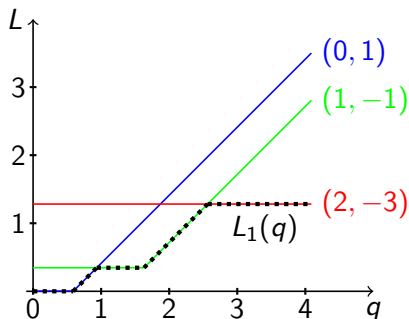


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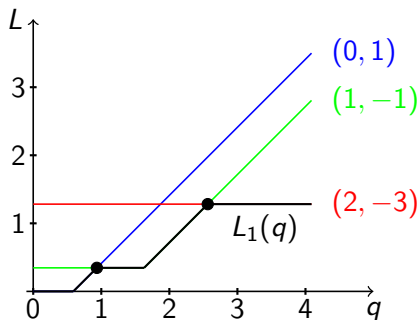
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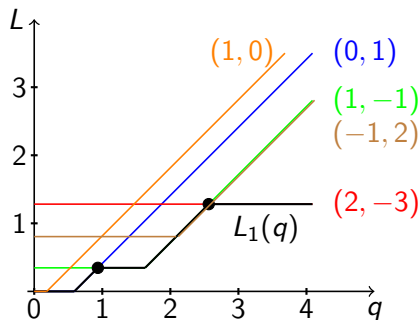
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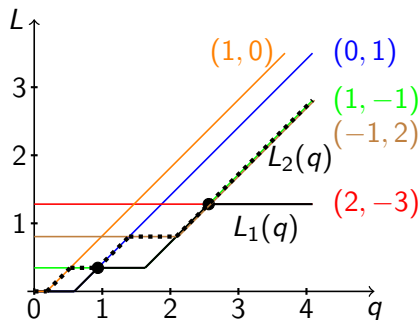
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The main constraint (general case)

- Write $\mathbb{R}^n = U \perp \langle \mathbf{u} \rangle_{\mathbb{R}}$ where $U = \mathbf{u}^{\perp}$,

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- $L_1^{(k)}(q) = L_1(q) + \cdots + L_k(q) + \mathcal{O}(1)$
is continuous and piecewise linear with slopes 0 and 1 on $[0, \infty)$,
- at each q where $L_1^{(k)}$ changes slope from 1 to 0, we have
 $L_{k+1}(q) = L_k(q) + \mathcal{O}(1)$,

because then $L_1^{(k)}(q) = L_2^{(k)}(q) = L_1(q) + \cdots + L_{k-1}(q) + L_{k+1}(q) + \mathcal{O}(1)$.

An approximation of $\mathbf{L}_u$

Schmidt and Summerer transpose all conditions in terms of

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$$\text{Define } P_k(q) := \begin{cases} L_1^{(1)}(q) & \text{if } k = 1, \\ L_1^{(k)}(q) - L_1^{(k-1)}(q) & \text{if } 1 \leq k \leq n-1, \\ q - L_1^{(n-1)}(q) & \text{if } k = n, \end{cases}$$

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Then, $\mathbf{P}$ is an (n, γ) -system for some $\gamma > 0$, according to the following definition.

(n, γ) -systems

Definition. An (n, γ) -system is a map $\mathbf{P} = (P_1, \dots, P_n): [0, \infty) \rightarrow \mathbb{R}^n$ which satisfies the following conditions.

- (S1) $P_j(q) \leq P_{j+1}(q) + \gamma \quad (1 \leq j < n, 0 \leq q).$
- (S2) $P_j(q_1) \leq P_j(q_2) + \gamma \quad (1 \leq j \leq n, 0 \leq q_1 \leq q_2).$
- (S3) For $j = 1, \dots, n$, the function $M_j := P_1 + \dots + P_j: [0, \infty) \rightarrow \mathbb{R}$ is continuous and piecewise linear with slopes 0 and 1.
- (S4) $M_n(q) = q \quad (0 \leq q).$
- (S5) If, for $j \in \{1, \dots, n-1\}$, the function M_j changes slope from 1 to 0 at a point $q > 0$, then $P_{j+1}(q) \leq P_j(q) + \gamma.$

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(S1) $P_j(q) \leq P_{j+1}(q) + \gamma \quad (1 \leq j < n, 0 \leq q).$

(S2) $P_j(q_1) \leq P_j(q_2) + \gamma \quad (1 \leq j \leq n, 0 \leq q_1 \leq q_2).$

(S3) For $j = 1, \dots, n$, the function $M_j := P_1 + \dots + P_j: [0, \infty) \rightarrow \mathbb{R}$ is continuous and piecewise linear with slopes 0 and 1.

(S4) $M_n(q) = q \quad (0 \leq q).$

(S5) If, for $j \in \{1, \dots, n-1\}$, the function M_j changes slope from 1 to 0 at a point $q > 0$, then $P_{j+1}(q) \leq P_j(q) + \gamma.$

(S3) and (S4) $\Rightarrow P_1$ and P_n have slopes 0 and 1

$P_2, \dots, P_{n-1}$ have slopes $-1, 0$ and 1

The theorem of Schmidt and Summerer

Theorem (Schmidt-Summerer, 2013)

Let $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\| = 1$. There exist $\gamma > 0$ and an (n, γ) -system $\mathbf{P}: [0, \infty) \rightarrow \mathbb{R}^n$ such that

$$\|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq \gamma \quad \text{for each } q \geq 0.$$

The authors use a different but equivalent setting.

Ideal case

Definition. An $(n, 0)$ -system is a map $\mathbf{P} = (P_1, \dots, P_n): [q_0, \infty) \rightarrow \mathbb{R}^n$ which satisfies the following conditions.

- (1) $P_j(q) \leq P_{j+1}(q) + 0 \quad (1 \leq j < n, q_0 \leq q).$
- (2) $P_j(q_1) \leq P_j(q_2) + 0 \quad (1 \leq j \leq n, q_0 \leq q_1 \leq q_2).$
- (3) For $j = 1, \dots, n$, the function $M_j := P_1 + \dots + P_j: [q_0, \infty) \rightarrow \mathbb{R}$ is continuous and piecewise linear with slopes 0 and 1.
- (4) $M_n(q) = q \quad (q_0 \leq q).$
- (5) If, for $j \in \{1, \dots, n-1\}$, the function M_j changes slope from 1 to 0 at a point $q > q_0$, then $P_{j+1}(q) \leq P_j(q) + 0.$

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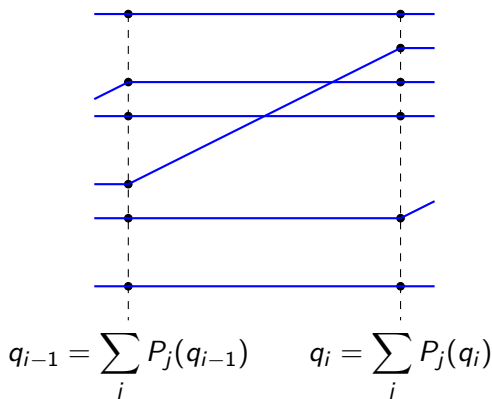
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- A switch number of $\mathbf{P}$ is q_0 or any point $q > q_0$ at which at least one of the functions $M_1, \dots, M_{n-1}$ changes slope from 0 to 1.
- Let $\delta > 0$. We say that $\mathbf{P}$ is *rigid with mesh δ* if, at each switch number q , the coordinates of $\mathbf{P}(q)$ form a strictly increasing sequence of positive integer multiples of δ .

The combined graph of a rigid system

Suppose that $\mathbf{P} = (P_1, \dots, P_n)$ is a rigid system with mesh δ on $[q_0, \infty)$ and let $q_0 < q_1 < \dots$ be its switch numbers.

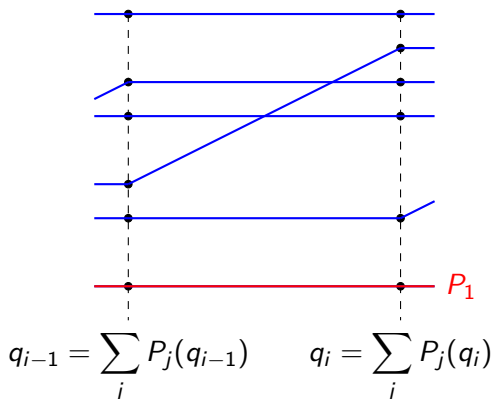
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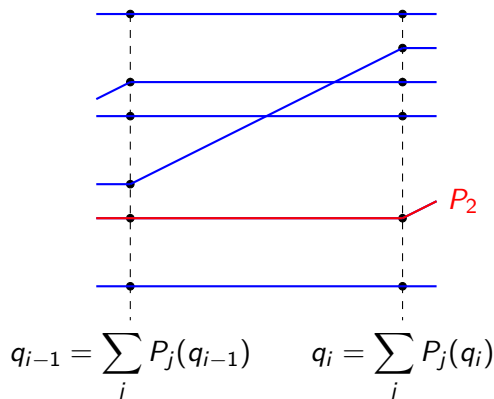
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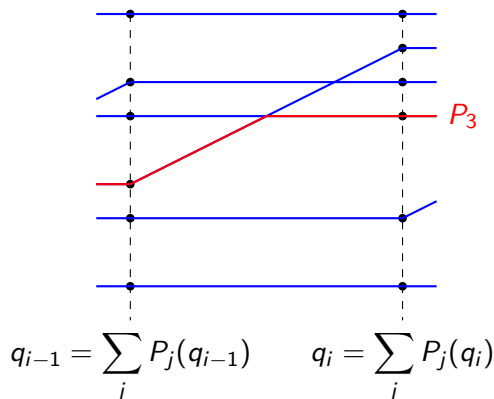
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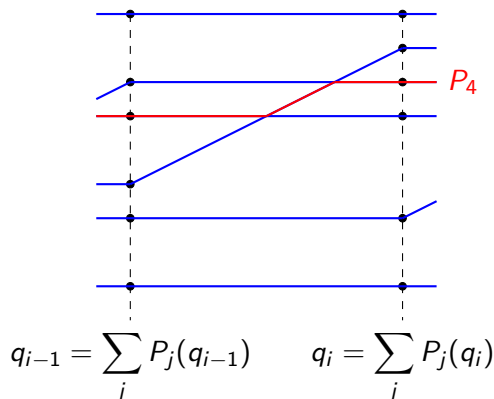
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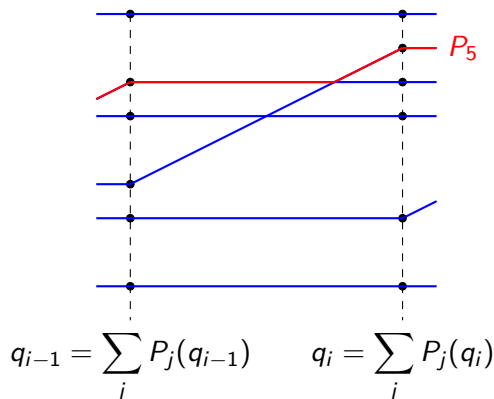
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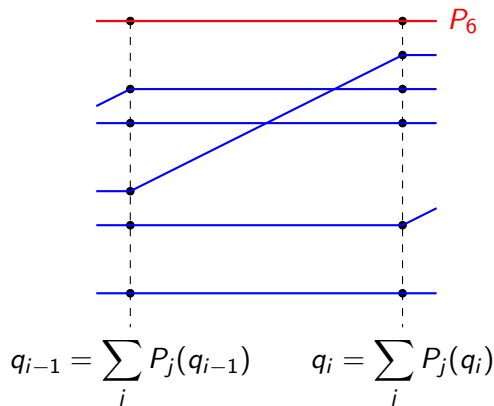
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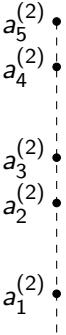
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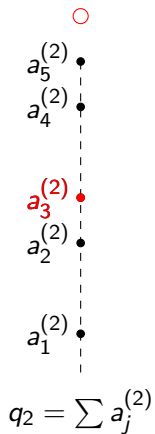
Construction of a rigid 5-system



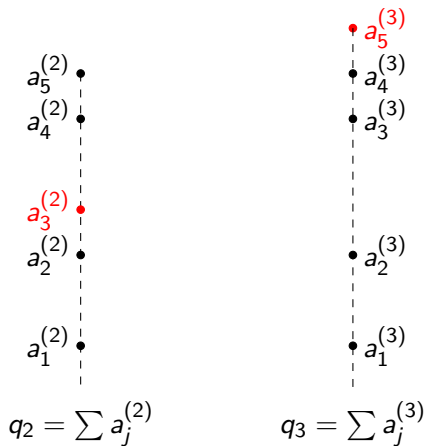
The diagram illustrates the construction of a rigid 5-system. It features a vertical dashed line with five points marked by solid black dots. From top to bottom, the points are labeled $a_5^{(2)}$, $a_4^{(2)}$, $a_3^{(2)}$, $a_2^{(2)}$, and $a_1^{(2)}$. Below the dashed line, the equation $q_2 = \sum a_j^{(2)}$ is written, indicating that the sum of the values at these points defines the coordinate q_2 .

$$q_2 = \sum a_j^{(2)}$$

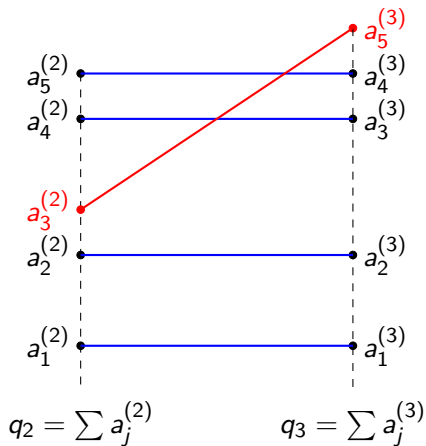
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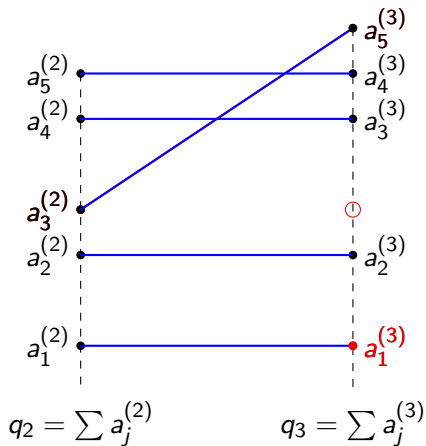
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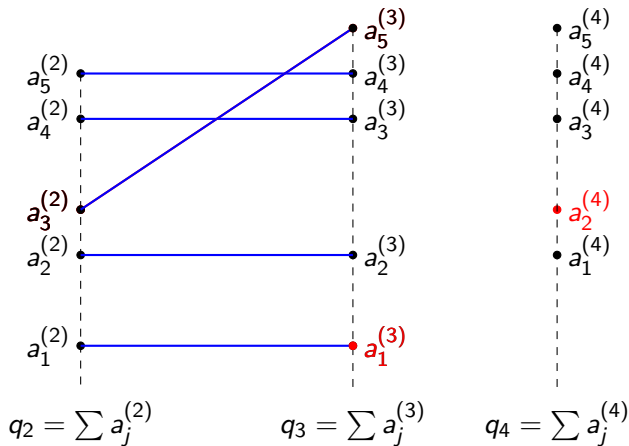
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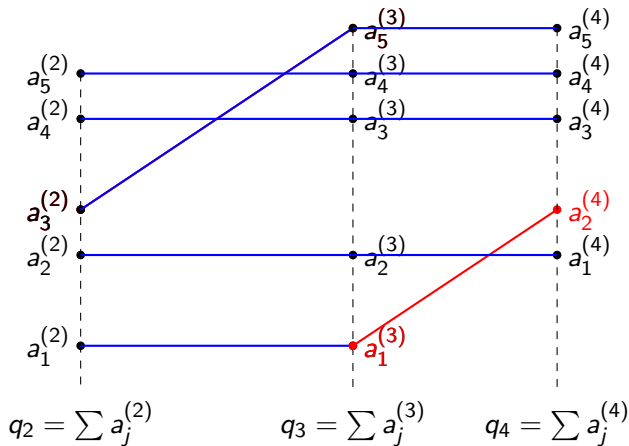
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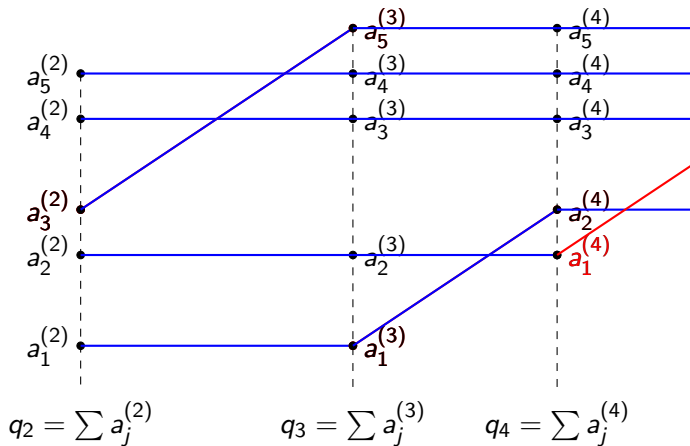
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Canvases (formal definition)

A rigid system of mesh δ is equivalent to the following data, which we call a *canevas*:

- an element $s \in \{1, 2, 3, \dots\} \cup \{\infty\}$,
- for each index i with $0 \leq i < s$, a pair of integers k_i and ℓ_i with
$$1 \leq k_i < \ell_i \leq n$$
- for each index i with $1 \leq i < s$, a point $\mathbf{a}^{(i)} = (a_1^{(i)}, \dots, a_n^{(i)})$ whose coordinates form a strictly increasing sequence of positive multiples of δ with

$$(a_1^{(i-1)}, \dots, \widehat{a_{k_{i-1}}^{(i-1)}}, \dots, a_n^{(i-1)}) = (a_1^{(i)}, \dots, \widehat{a_{\ell_i}^{(i)}}, \dots, a_n^{(i)})$$

and $a_{\ell_i}^{(i)} > a_{k_{i-1}}^{(i-1)}$.

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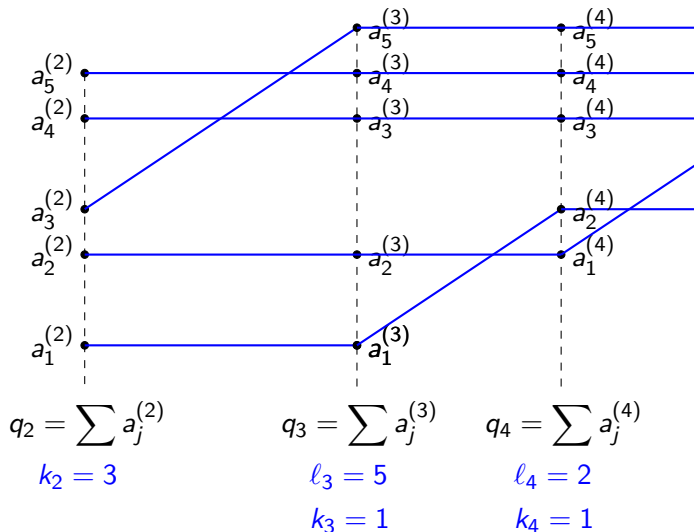
$$\text{and } a_{\ell_i}^{(i)} > a_{k_{i-1}}^{(i-1)}.$$

Just set

$$q_i = a_1^{(i)} + \dots + a_n^{(i)} \quad (0 \leq i < s)$$

and connect each $(q_{i-1}, a_j^{(i-1)})$ with the appropriate $(q_i, a_h^{(i)})$.

Example



III: Main results

A combinatorial argument shows:

Theorem A (2015)

Fix $\gamma, \delta > 0$, and let $\tilde{\mathbf{P}}: [0, \infty) \rightarrow \mathbb{R}^n$ be an (n, γ) -system. There exists a rigid system $\mathbf{P}: [q_0, \infty) \rightarrow \mathbb{R}^n$ with mesh δ such that $\tilde{\mathbf{P}} - \mathbf{P}$ is bounded.

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In view of the theorem of Schmidt and Summerer, this gives:

Corollary

Let $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\| = 1$. For each $\delta > 0$ there exists a rigid system $\mathbf{P}: [q_0, \infty) \rightarrow \mathbb{R}^n$ with mesh δ such that $\mathbf{P} - \mathbf{L}_{\mathbf{u}}$ is bounded.

Towards the converse

Let $\mathbf{P}: [q_0, \infty) \rightarrow \mathbb{R}^n$ be a rigid system with mesh $\delta \geq 5n\epsilon$ where $\epsilon = 2^n \log(n)$. Consider its associated sequences

$$(\mathbf{a}^{(i)})_{0 \leq i < s}, \quad (k_i)_{0 \leq i < s}, \quad (\ell_i)_{0 \leq i < s} \quad \text{and} \quad (q_i)_{0 \leq i < s}.$$

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- *above each $q \in [q_i, q_{i+1})$, the trajectories of $\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)}$ are at distance at most ϵ from $\mathbf{P}(q)$.*

Illustration ($n = 5$)

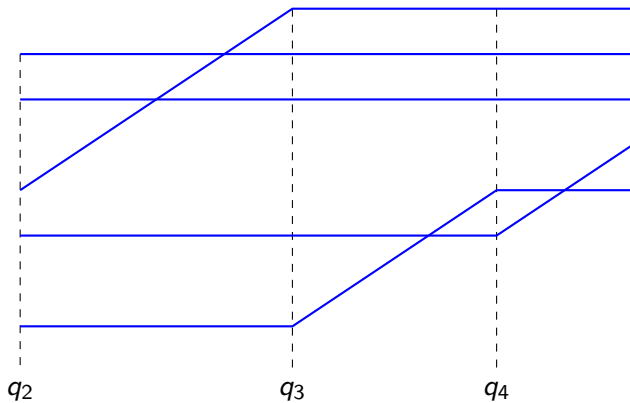


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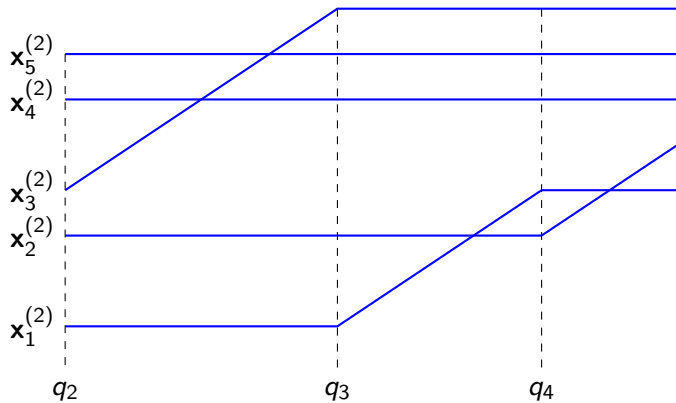
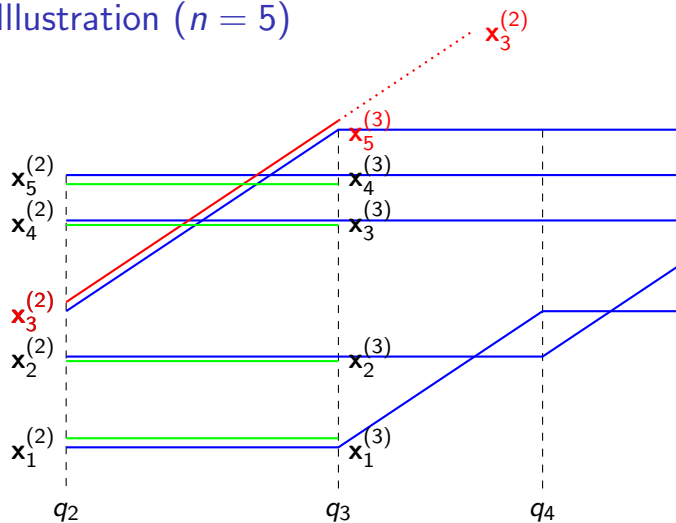
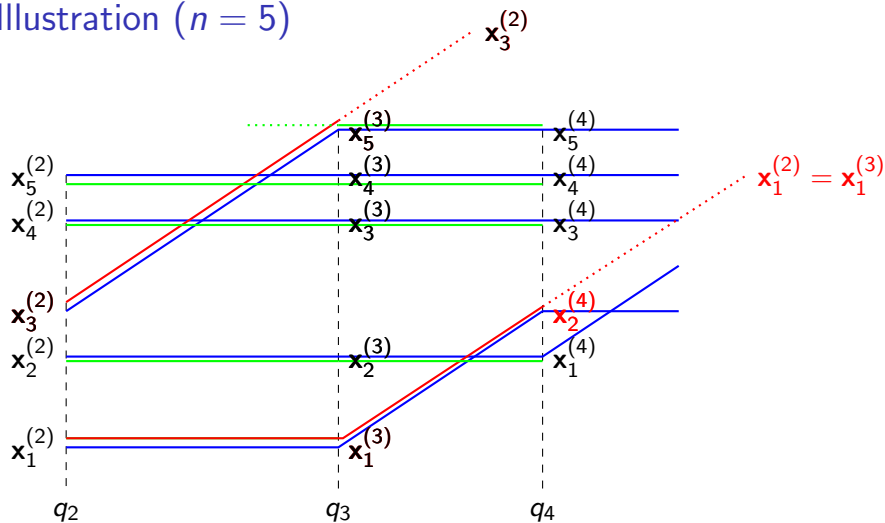


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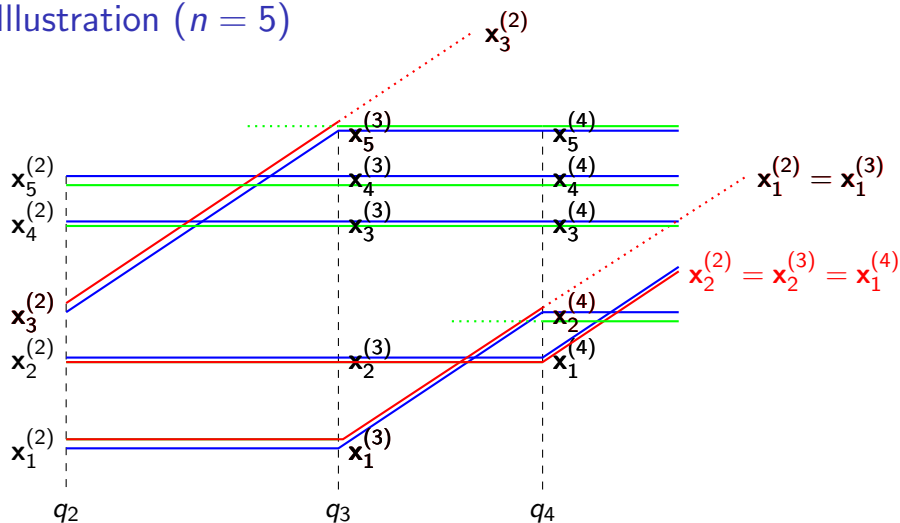
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For each sufficiently large $\delta > 0$ and each rigid system $P: [q_0, \infty) \rightarrow \mathbb{R}^n$ with mesh δ , there exists a unit vector $\mathbf{u} \in \mathbb{R}^n$ such that $\mathbf{P} - \mathbf{L}_{\mathbf{u}}$ is bounded.

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We take $\mathbf{u}$ to be the limit of the unit vectors perpendicular to the spaces spanned by the latter $(n - 1)$ -tuples.

IV: Applications

Endow $\mathbb{R}^n$ with the component-wise ordering:

$$(a_1, \dots, a_n) \leq (b_1, \dots, b_n) \iff a_i \leq b_i \text{ for } i = 1, \dots, n.$$

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 $\mathbf{x} \longmapsto (T_1(\mathbf{x}), \dots, T_m(\mathbf{x}))$

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- The spectrum of T is

$$\{\mu_T(\mathbf{L}_{\mathbf{u}}); \mathbf{u} \in \mathbb{R}^n \text{ has } \mathbb{Q}\text{-linearly independent coordinates}\}$$

Example

Define $T: \mathbb{R}^n \rightarrow \mathbb{R}^{2n}$ by $T(\mathbf{a}) = (\mathbf{a}, -\mathbf{a})$.

For a nonzero $\mathbf{u} \in \mathbb{R}^n$, we have

$$\begin{aligned}\mu_T(\mathbf{L}_{\mathbf{u}}) &= \liminf_{q \rightarrow \infty} \left(\frac{L_{\mathbf{u},1}(q)}{q}, \dots, \frac{L_{\mathbf{u},n}(q)}{q}, -\frac{L_{\mathbf{u},1}(q)}{q}, \dots, -\frac{L_{\mathbf{u},n}(q)}{q} \right) \\ &= (\varphi_1(\mathbf{u}), \dots, \varphi_n(\mathbf{u}), -\bar{\varphi}_1(\mathbf{u}), \dots, -\bar{\varphi}_n(\mathbf{u}))\end{aligned}$$

So the spectrum of T is the same as that of $(\varphi_1, \dots, \varphi_n, -\bar{\varphi}_1, \dots, -\bar{\varphi}_n)$.

Some known spectra

- $(\underline{\varphi}_1, \bar{\varphi}_n)$: Khintchine (1926, 1928), Jarník (1935, 1936)
- $(\bar{\varphi}_1, \underline{\varphi}_n)$: Jarník (1938), German (2012), Schmidt and Summerer (2016), Marnat (≥ 2017)
- $(\underline{\varphi}_1, \bar{\varphi}_1, \underline{\varphi}_3, \bar{\varphi}_3)$ for $n = 3$: Laurent (2009)
- $(\underline{\psi}_1, \dots, \underline{\psi}_{n-1})$: Schmidt (1967), Laurent (2009), Roy (2016)

where

$$\underline{\psi}_i(\mathbf{u}) = \liminf_{q \rightarrow \infty} q^{-1}(L_{\mathbf{u},1}(q) + \dots + L_{\mathbf{u},i}(q)) \quad (1 \leq i \leq n-1).$$

Link with parametric geometry of numbers

Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map.

Choose a nonzero $\mathbf{u} \in \mathbb{R}^n$ and an n -system $\mathbf{P} = (P_1, \dots, P_n)$ such that $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded.

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Therefore the spectrum of T is also given by

$$\{\mu_T(\mathbf{P}); \mathbf{P} \text{ is a proper } n\text{-system}\}$$

Main result

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. Then

- The set

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- The spectrum of T is a **compact connected subset** of $\mathbb{R}^m$.
- If $n = 3$, then the spectrum of T is a **semi-algebraic subset** of $\mathbb{R}^m$.
Moreover, it is closed under componentwise minimum: if $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ belong to the spectrum of T , then

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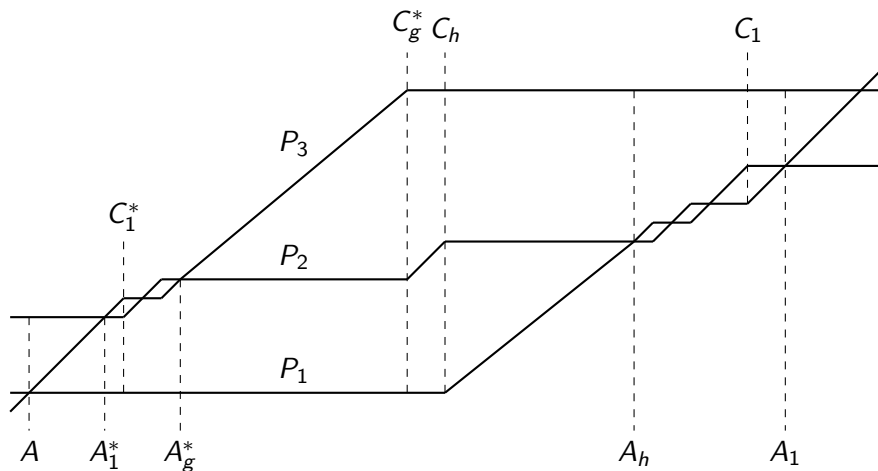
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Application: Computation of the spectrum of $(\varphi_1, \bar{\varphi}_1, \varphi_2, \bar{\varphi}_2, \varphi_3, \bar{\varphi}_3)$ when $n = 3$.

Basic pattern for the combined graph of a 3-system



Corresponding pattern for the image of $q \mapsto q^{-1}\mathbf{P}(q)$

