

Parametric geometry of numbers in function fields

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- transparencies: <http://aix1.uottawa.ca/~droy>

Parametric geometry of numbers over $\mathbb{R}$

Fix $\mathbf{u} \in \mathbb{R}^n$ with Euclidean norm $\|\mathbf{u}\| = 1$. Define

$$\mathcal{C}_{\mathbf{u}}(e^q) = \{\mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| \leq 1 \text{ and } |\mathbf{u} \cdot \mathbf{x}| \leq e^{-q}\} \quad (q \geq 0).$$

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For $i = 1, \dots, n$, set

$L_{\mathbf{u},i}(q)$ = the smallest $t \geq 0$ such that $e^t \mathcal{C}_{\mathbf{u}}(e^q)$ contains at least i linearly independent points of $\mathbb{Z}^n$

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Schmidt (1982) proposes to study the map

$$\begin{aligned} \mathbf{L}_{\mathbf{u}}: [0, \infty) &\longrightarrow \mathbb{R}^n \\ q &\longmapsto (L_{\mathbf{u},1}(q), \dots, L_{\mathbf{u},n}(q)) \end{aligned}$$

Characterization up to bounded functions

Recall: $L_{\mathbf{u},i}(q)$ = the smallest $t \geq 0$ for which the solutions $\mathbf{x} \in \mathbb{Z}^n$ of

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- Schmidt (1982) notes that
 - ▶ $0 \leq L_{\mathbf{u},1}(q) \leq \cdots \leq L_{\mathbf{u},n}(q)$ and $L_{\mathbf{u},1}(q) + \cdots + L_{\mathbf{u},n}(q) = q + \mathcal{O}(1)$,
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 - ▶ each $L_{\mathbf{u},j}$ is continuous and piecewise linear with slopes 0 and 1.
- Schmidt and Summerer (2013) show that $\mathbf{L}_{\mathbf{u}} = (L_{\mathbf{u},1}, \dots, L_{\mathbf{u},n})$ can be approximated up to a bounded function by “an (n, γ) -system” for some $\gamma > 0$.

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- R. (2015): for each unit vector $\mathbf{u} \in \mathbb{R}^n$, there is an $(n, 0)$ -system $\mathbf{P}$ for which $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded and conversely, for each $(n, 0)$ -system $\mathbf{P}$ there is a unit vector $\mathbf{u} \in \mathbb{R}^n$ with the same property.

n -systems

Definition (Schmidt and Summerer 2013)

An $(n, 0)$ -system, or simply n -system, is a map

$$\begin{aligned} \mathbf{P}: [0, \infty) &\longrightarrow \mathbb{R}^n \\ q &\longmapsto (P_1(q), \dots, P_n(q)) \end{aligned}$$

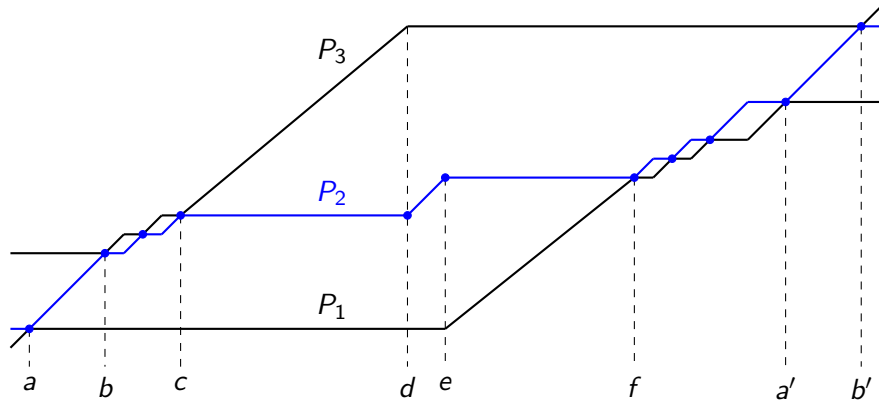
such that

- $0 \leq P_1(q) \leq \dots \leq P_n(q)$ and $P_1(q) + \dots + P_n(q) = q$ ($q \geq 0$),
- each P_i is continuous and piecewise linear with slopes 0 and 1,
- for each $q > 0$, there is exactly one pair of indices k and ℓ for which

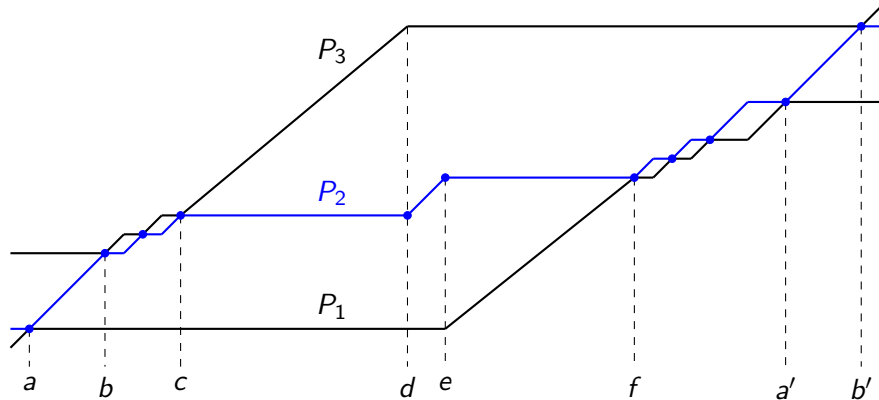
$$P'_k(q^+) = 1 \quad \text{and} \quad P'_\ell(q^-) = 1,$$

- whenever $\ell < k$, we have $P_\ell(q) = \dots = P_k(q)$.

Basic pattern for the combined graph of a 3-system

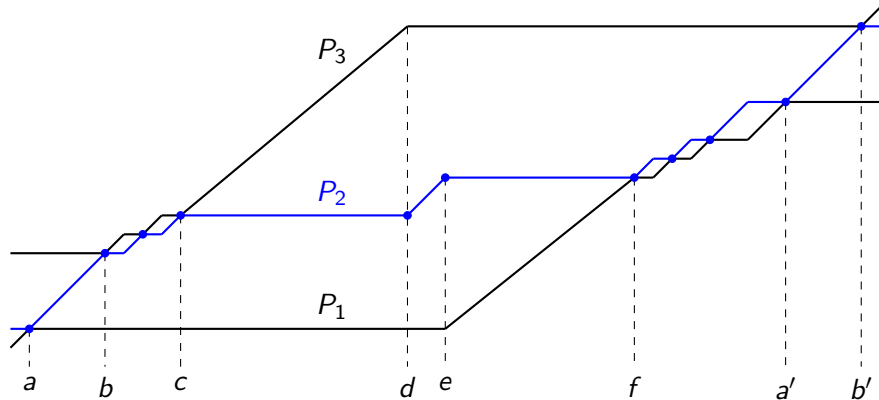


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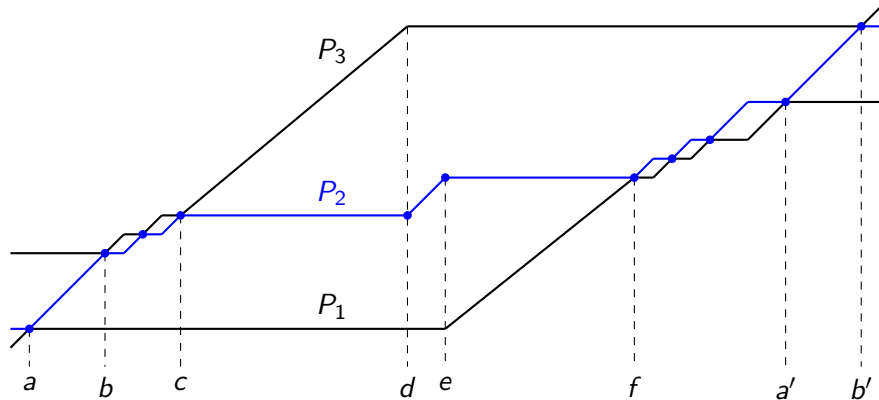
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- at d , we have $\ell = 3$ and $k = 2$, no constraint.

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Let $\mathbf{u} \in F((1/T))^n$ with $\|\mathbf{u}\| = 1$. For each $i = 1, \dots, n$ and $q \geq 0$, set

- $L_{\mathbf{u},i}(q) =$ the smallest $t \geq 0$ for which the solutions $\mathbf{x} \in F[T]^n$ of

$$\|\mathbf{x}\| \leq e^t \quad \text{and} \quad |\mathbf{u} \cdot \mathbf{x}| = \left| \sum u_j x_j \right| \leq e^{t-q}$$

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- $\mathbf{L}_{\mathbf{u}}: [0, \infty) \longrightarrow \mathbb{R}^n$.
$$q \longmapsto (L_{\mathbf{u},1}(q), \dots, L_{\mathbf{u},n}(q))$$

Characterization of the maps L_u

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Theorem

The set of maps $\mathbf{L}_{\mathbf{u}}$ with $\mathbf{u} \in F((1/T))^n$ of norm 1 is the same as the set of all n -systems $\mathbf{P}: [0, \infty) \rightarrow \mathbb{R}^n$ such that $\mathbf{P}(q) \in \mathbb{N}^n$ for each $q \in \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \dots\}$.

We call them *integral n -systems*.

Canvases

There is a bijection between the set of integral n -systems and the set of finite or infinite sequences of integer n -tuples

$$\mathbf{a}^{(i)} = (a_1^{(i)}, \dots, a_n^{(i)}) \quad (0 \leq i < s)$$

equipped with sequences of indices k_i and ℓ_i ($0 \leq i < s$) such that:

- $\mathbf{a}^{(0)} = (0, \dots, 0)$, $k_0 = \ell_0 = n$,
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Example of such a sequence (canvas) of integer triples:

$$(0, 0, \underline{0}), \quad (0, 0, \overline{3}),$$

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The combined graph of the system attached to a canvas

For each index i with $0 \leq i < s$, draw points with abscissa

$$q_i = a_1^{(i)} + \cdots + a_n^{(i)}$$

and ordinates $a_1^{(i)}, \dots, a_n^{(i)}$. Then link the points using line segments of slope 0 and 1.

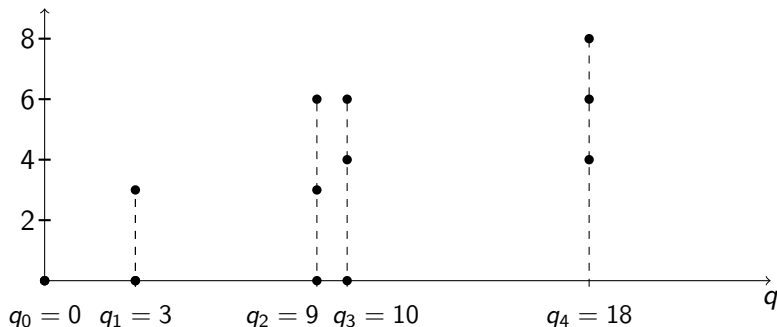
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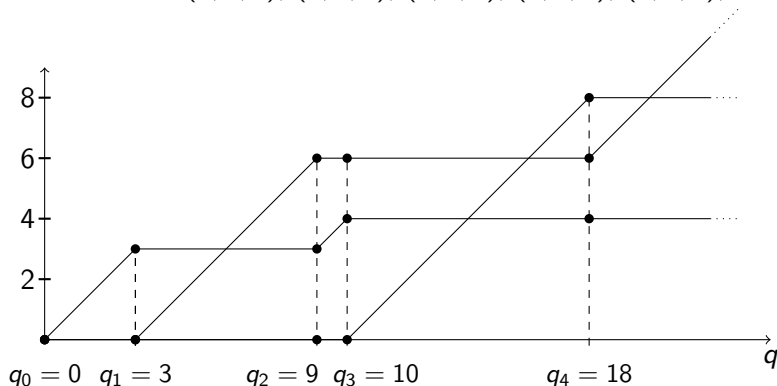
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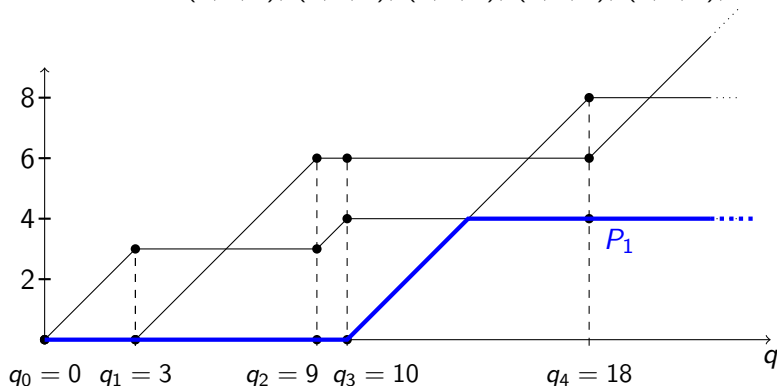
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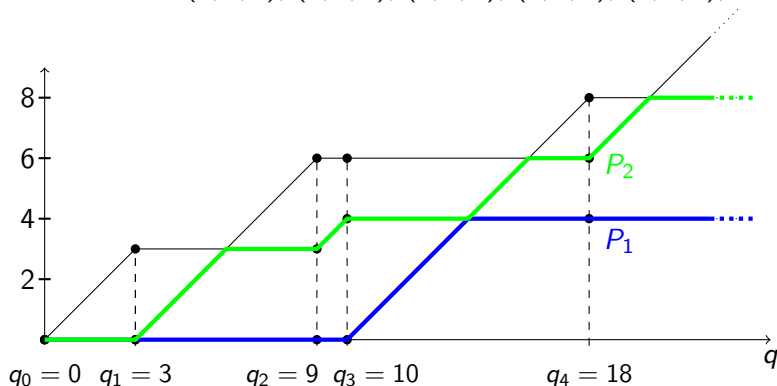
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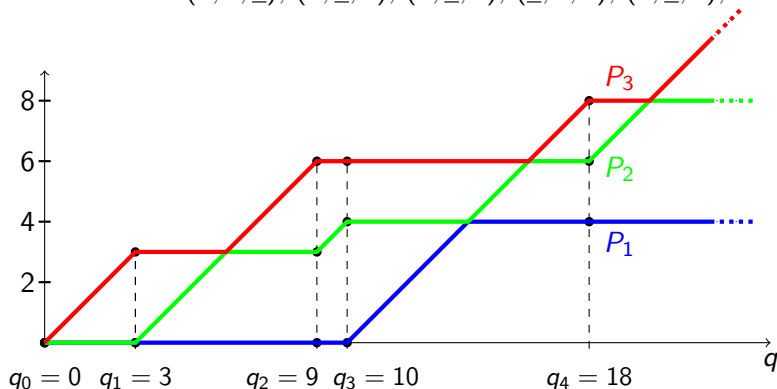
The combined graph of the system attached to a canvas

For each index i with $0 \leq i < s$, draw points with abscissa

$$q_i = a_1^{(i)} + \cdots + a_n^{(i)}$$

and ordinates $a_1^{(i)}, \dots, a_n^{(i)}$. Then link the points using line segments of slope 0 and 1.

For the canvas $(0, 0, \underline{0})$, $(0, \underline{0}, \bar{3})$, $(0, \underline{3}, \bar{6})$, $(\underline{0}, \bar{4}, 6)$, $(4, \underline{6}, \bar{8})$, ... we find:



Construction of the point $\mathbf{u}$

We construct recursively a sequence of bases

$$(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)}) \text{ of } F[T]^n \quad (0 \leq i < s)$$

so that:

- $(\mathbf{x}_1^{(0)}, \dots, \mathbf{x}_n^{(0)}) = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ is the canonical basis of $F[T]^n$,
- whenever $1 \leq i < s$, we have
 - ▶ $(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)}) = (\mathbf{x}_1^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_n^{(i-1)})$
 - ▶ $\mathbf{x}_{\ell_i}^{(i)} = \mathbf{x}_{k_{i-1}}^{(i-1)} + T^b \mathbf{x}_{k_i}^{(i)}$ where $b = a_{\ell_i}^{(i)} - a_{k_i}^{(i)}$.

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Define $\mathbf{u}^{(i)} \in F((1/T))^n$ as the solution of the system

$$\mathbf{u}^{(i)} \cdot \mathbf{x}_j^{(i)} = 0 \quad \text{for } j \neq k_i \quad \text{and} \quad \mathbf{u}^{(i)} \cdot \mathbf{e}_n = 1.$$

Then $(\mathbf{u}^{(i)})_{0 \leq i < s}$ converges to a point $\mathbf{u} \in F((1/T))^n$ of norm 1 such that $\mathbf{L}_{\mathbf{u}}$ is the n -system $\mathbf{P}$ attached to the canvas (if $s < \infty$, take $\mathbf{u} = \mathbf{u}^{(s-1)}$).

The simplest integral n -system

Any integral n -system $\mathbf{P} = (P_1, \dots, P_n)$ satisfies

$$P_1(q) \leq \left\lfloor \frac{q}{n} \right\rfloor \leq \left\lceil \frac{q}{n} \right\rceil \leq P_n(q) \quad \text{for each } q \in \mathbb{N}.$$

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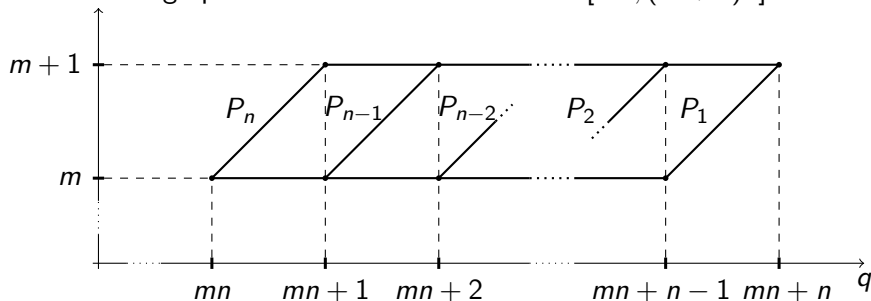
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Its combined graph over an interval of the form $[mn, (m+1)n]$ is:



Perfect systems

Theorem

Suppose that $\text{char}(F) = 0$. Let $\omega_1, \dots, \omega_n \in F$ be distinct, and let

$$\mathbf{u} = \left(e^{\omega_1/T}, \dots, e^{\omega_n/T} \right) \text{ where } e^{\omega/T} = \sum_{j=0}^{\infty} \frac{\omega^j}{j!} T^{-j} \in F[[1/T]] \quad (\omega \in F).$$

Then, we have $\|\mathbf{u}\| = 1$ and $\mathbf{L}_{\mathbf{u}}$ is the simplest integral n -system.

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Proof (Hermite, Mahler)

The series $f_1(T) = e^{\omega_1 T}, \dots, f_n(T) = e^{\omega_n T}$ in $F[[T]]$ form a perfect system:

For any nonzero $(\varrho_1, \dots, \varrho_n) \in \mathbb{N}^n$ and any nonzero $\mathbf{a} = (a_1, \dots, a_n) \in F[T]^n$ with

$$\deg(a_i) \leq \varrho_i - 1 \quad (1 \leq i \leq n) \quad \text{and} \quad \text{ord}_0\left(\sum a_i f_i\right) \geq \varrho_1 + \dots + \varrho_n - 1,$$

we have $\text{ord}_0\left(\sum a_i f_i\right) = \varrho_1 + \dots + \varrho_n - 1$.

Other examples

In fact we just need the condition for the diagonal n -tuples $(\varrho_1, \dots, \varrho_n)$ with $\varrho_1 = \dots = \varrho_n \neq 0$. So the same is true of

- $\mathbf{u} = \left((1 + 1/T)^{\omega_1}, \dots, (1 + 1/T)^{\omega_n} \right)$, where

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when $\omega_1, \dots, \omega_n \in F$ are pairwise incongruent modulo $\mathbb{Z}$;

- $\mathbf{u} = \left((\log(1 - 1/T))^{n-1}, \dots, \log(1 - 1/T), 1 \right)$ where

$$\log(1 - T) = - \sum_{j=1}^{\infty} \frac{T^j}{j}.$$

Reference: H. Jager, 1964.

A generalization of a result of Baker

We work with $F = \mathbb{C}$.

- for any $f = c_m T^m + c_{m+1} T^{m+1} + \dots \in \mathbb{C}((T))$ with $c_m \neq 0$, set $|f|_0 = e^{-m}$,
- for any $f = c_m T^{-m} + c_{m+1} T^{-m-1} + \dots \in \mathbb{C}((1/T))$ with $c_m \neq 0$, set $|f|_\infty = e^{-m}$.

Proposition

Let $\mathbf{u} = (e^{\omega_1/T}, \dots, e^{\omega_n/T}) \in \mathbb{C}[[1/T]]^n$ where $\omega_1, \dots, \omega_n \in \mathbb{C}$ are distinct. For any n -tuple of nonzero polynomials $\mathbf{a} = (a_1(T), \dots, a_n(T))$ in $\mathbb{C}[T]$, we have

$$|a_1|_0 \cdots |a_n|_0 |a_1|_\infty \cdots |a_n|_\infty |\mathbf{a} \cdot \mathbf{u}|_\infty \geq C(n)^{-1} \|\mathbf{a}\|_\infty,$$

where $C(n) = \exp(n(n-1)/2)$.

Corollary (Baker (1964))

For any nonzero $a_1(T), \dots, a_n(T) \in \mathbb{C}[T]$, we have

$$\left| a_1(T)e^{\omega_1/T} + \dots + a_n(T)e^{\omega_n/T} \right|_{\infty} \prod_{i=2}^n |a_i(T)|_{\infty} \geq C(n)^{-1},$$

$$|a_1(T)|_{\infty} \prod_{i=2}^n \left| a_1(T)e^{\omega_i/T} - a_i(T)e^{\omega_1/T} \right|_{\infty} \geq C(n)^{-(n-1)}.$$